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On the Optimal Choice of Nodes in the Collocation-Projection Method for Solving Linear Operator Equations

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$\mathscr{H}_R$ is a Hilbert space with norm $\|\ \|_R$ and K is a linear operator mapping $\mathscr{H}_R$ into $\mathscr{L}_2(T)$, where T is a closed, bounded interval of the real line. K and $\mathscr{H}_R$ jointly have the property that there exist M such that $|(Kf)(t)| \leqslant M\|f\|_R$, $t \in T$. In the collocation–projection method for approximately solving the equation $Kf = g$, the approximate solution f_n is taken as that element of $\mathscr{H}_R$ that minimizes $\|f\|_R$ subject to $(Kf)(t) = g(t)$ for $t \in T_n$, where $T_{n+1} = \{t_i\}_{i=0}^n$ and $t_i \in T$. We show how results obtained elsewhere are applicable to the problem of choosing the node set T_n to minimize $\|f - f_n\|_R$. The results are applicable to the approximate solution of 2-point boundary value problems using a spline basis. We make some remarks concerning the problem of choosing the nodes to minimize $\sup_s |f(s) - f_n(s)|$, when K is a differential operator.

1. INTRODUCTION

Let $\mathscr{H} = W_2^{(r)}$ be the Hilbert space of real valued functions on $[0, 1]$:

$$\mathscr{H} = \{f: f^{(\nu)} \text{ abs cont}, \qquad \nu = 0, 1,..., r-1, \qquad f^{(r)} \in \mathscr{L}_2[0, 1]\},$$

endowed with a suitable norm, for example

$$\|f\|^2 = \sum_{\nu=0}^{r-1} (f^{(\nu)}(0))^2 + \int_0^1 [f^{(r)}(u)]^2\, du.$$

Let K be a pth-order linear differential operator with smooth coefficients (and given boundary conditions $\mathscr{B}$), $p < r$. An approximate solution f_n to the linear operator equation

$$(Kf)(t) = g(t), \qquad t \in T, f \in \mathscr{B} \tag{1.1}$$

is obtained by letting f_n be the solution to the problem: Find $f \in \mathscr{H} \cap \mathscr{B}$ minimizing $\|f\|$ subject to

$$(Kf)(t) = g(t), \qquad t \in T_n,$$

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where

$$T_n = \{t_i\}_{i=0}^n$$

and $t_0 < t_1 < \cdots < t_n$, $T = [t_0, t_n]$. This method for solving (1.1) approximately is discussed in Athavale [1], Golomb [7], and Wahba [19] and convergence rates for $\|f - f_n\|$ and $|f(s) - f_n(s)|$ under various hypotheses are given in [7, 19]. An explicit (computable) formula for f_n may be obtained by using the fact that the linear functional that maps f into $(Kf)(t)$ is continuous in $\mathscr{H}$ (for each t) and thus, has a representer η_t, such that

$$(Kf)(t) = \langle \eta_t, f \rangle_{\mathscr{H}}.$$

Thus, f_n is in the span of the union of $\{\eta_t, t \in T_n\}$ and the representers of the boundary functionals. This subspace is also spanned by a certain family of B-splines (see [1, 7] for details), so that the method is a collocation method on a B-spline basis and hence, similar to, e.g., that discussed by deBoor and Swartz [5]. The knots of the $\mathscr{B}$ splines coincide with the collocation node set T_n.

In this paper, we adress the question of how to choose the node set T_n to minimize $\|f - f_n\|$. What we do here is to obtain a density function h^*, describing, approximately, for large n, the distribution of the node set $\{t_{in}\}_{i=0}^n$ that minimizes $\|f - f_n\|$. Actually, we are able to do this for a much larger class of linear operator equations than boundary values problems and we describe this class shortly.

The problem, in the generality in which we consider it, is actualy isomorphic to the regression design problem of Sacks and Ylvisaker, studied extensively in the context of experimental design in the statistical literature, see Hajek and Kimeldorf [8], Sacks and Ylvisaker [12–15], and Wahba [17, 20]. It is also isomorphic to certain of the nonlinear approximation problems studied by Karlin [9, 10] that were originally motivated by the problem of optimal quadrature formulae posed by Schoenberg [16].

Here, we demonstrate how the results of [20] may be carried over to choosing h^*, the approximate distribution of the optimal node set, in a fairly large class of approximate solutions to linear operator equations including the boundary value problem considered above. We do not present a proof of the main theorem on the determination of h^* here, because the proof (which is long and difficult) appears (in the experimental design context) in [20]. Our purpose here is to show how the results of [20] (and the earlier work [8, 12–15, 17]) apply to the approximate solution of linear operator equations. For other results on this problem and a bibliography, see deBoor [6], also deBoor and Swartz [5], Burchard [2], and Burchard and Hale [3].

The solution density h^* depends, of course, on properties of f that are not generally known, a priori. A two stage numerical algorithm has been developed in [1] for boundary value problems, where, in the first stage, information is obtained that can be used to select an approximately optimal set in the second stage according to the results given here. We also discuss the choice of T_n to minimize $\sup_s |f(s) - f_n(s)|$. The results are not in a form practical for computation, however.

We now describe the more general class of collocation–projection methods for finding approximate solutions to linear operator equations, to which our results apply. These methods are described in detail in Wahba [18, 19]. Applications to integral, differential, and integro-differential equations are given there, along with convergence rates. We let

$$Kf = g, \tag{1.2}$$

where $f \in \mathscr{H}_R$, $\mathscr{H}_R$ being a Hilbert space of real valued functions on a closed bounded interval S of the real line. We assume that $\mathscr{H}_R$ possesses a reproducing kernel, $R(s, s')$, $s, s' \in S$. K maps $\mathscr{H}_R$ into $\mathscr{L}_2[T]$, where T is a closed bounded interval of the real line. For our results, we only require that K and $\mathscr{H}_R$ jointly have the property that the linear functional that maps $f \in \mathscr{H}_R$ to $(Kf)(t)$ is a continuous linear functional on $\mathscr{H}_R$ for each $t \in T$. Denoting the inner product in $\mathscr{H}_R$ by $\langle\ ,\ \rangle_R$, it follows that for each $t \in T$, there exists $\eta_t \in \mathscr{H}_R$ such that

$$(Kf)(t) = \langle \eta_t, f \rangle_R, \qquad f \in \mathscr{H}_R.$$

The approximate solution to (1.2) is taken as the solution f_n to the problem: Find $f \in \mathscr{H}_R$ to minimize $\|f\|_R$ subject to

$$(Kf)(t) = g(t), \qquad t \in T_n.$$

The solution f_n is the orthogonal projection, in $\mathscr{H}_R$, of f onto the subspace V_n spanned by $\{\eta_t, t \in T_n\}$ (with the obvious modification when boundary conditions on f are imposed). Thus, the method is simultaneously a collocation method and an orthogonal–projection method in a Hilbert space and any method simultaneously enjoying these properties must be of the type described here.

We let P_{V_n} denote the projection operator onto V_n and in the remainder of this paper we use the notation $f_n = P_{V_n} f$. Supposing that the set $\{\eta_t, t \in T_n\}$ is linearly independent in $\mathscr{H}_R$, we have

$$P_{V_n} f = (\eta_{t_1}, \eta_{t_2}, ..., \eta_{t_n}) Q_n^{-1} (g(t_1), ..., g(t_n))',$$

where Q_n is the $n \times n$ matrix with i, jth entry $Q(t_i, t_j)$, where $Q(t, t') = \langle \eta_t, \eta_{t'} \rangle_R$. Since

$$\eta_t(s) = (KR_s)(t)$$

and

$$Q(t, t') = (K\eta_{t'})(t),$$

where $R_s(s') = R(s, s')$; η_t and Q are generally known once R is given and $(P_{V_n}f)(s)$ is computable. We remark parenthetically that the grammian matrix Q_n is typically very poorly conditioned, also typically, at least when K is a differential operator, a basis for V_n with a decently conditioned grammian can be found, see [1]. Examples of reproducing kernels R for spaces topologically equivalent to $W_2^{(r)}$ may be found, e.g., in deBoor and Lynch [4], Kimeldorf and Wahba [11].

In this paper, we exhibit a density function h^* that describes, approximately, the distribution of the node set $\{t_{in}\}_{i=0}^n$ that minimizes $\|f - P_{V_n}f\|_R$, for a large class of cases where $K(\mathscr{H}_R)$ can be made into a Hilbert space topologically equivalent to $W_2^{(m)}$ for some m, and f satisfies some regularity conditions that basically eliminate those cases where an optimal node set by our criteria is either obtained trivially or is nonexistant.

2. The Optimal Choice of T_n to Minimize $\|f - P_{V_n}f\|_R$

It is well known [18] that $K(\mathscr{H}_R) = \mathscr{H}_Q$, where $\mathscr{H}_Q$ is the reproducing kernel Hilbert space (RKHS) with RK $Q(t, t')$ given by

$$Q(t, t') = \langle \eta_t, \eta_{t'} \rangle_R, \quad t, t' \in T.$$

Let V be the closure of the span of $\{\eta_t, t \in T\}$ in $\mathscr{H}_R$. Then, $V^\perp$ is the null space of K in $\mathscr{H}_R$. There is [18, Lemma 1] an isometric isomorphism between $\mathscr{H}_Q$ and V generated by the correspondence "$\sim$"

$$Q_t \in \mathscr{H}_Q \sim \eta_t \in V, \qquad t \in T, \tag{2.1}$$

where Q_t is the representer of the evaluation functional at t in $\mathscr{H}_Q$, $Q_t(s) = Q(s, t)$ and

$$g \in \mathscr{H}_Q \sim f \in V \Leftrightarrow g = Kf.$$

Thus, letting P_{T_n} be the projection operator in $\mathscr{H}_Q$ onto span$\{Q_t, t \in T_n\}$, $\| \, \|_Q$ be the norm in $\mathscr{H}_Q$, and P_V be the projection operator in $\mathscr{H}_R$ onto V,

$$\| P_V f - P_{V_n} f \|_R^2 = \| g - P_{T_n} g \|_Q^2,$$

$$\begin{aligned} \| f - P_{V_n} f \|_R^2 &= \| f - P_V f \|_R^2 + \| P_V f - P_{V_n} f \|_R^2 \\ &= \| f - P_V f \|_R^2 + \| g - P_{T_n} g \|_Q^2 . \end{aligned}$$

Hence, the problem reduces to that of finding T_n to minimize

$$\| g - P_{T_n} g \|_Q^2 .$$

If $V^\perp = \{0\}$ and $g = \sum_{i=0}^k c_i Q_{s_i}$ for some $\{c_i\}_{i=1}^k$ and $\{s_i\}_{i=1}^k$ and T_n contains $\{s_i\}_{i=1}^k$, then $\| g - P_{T_n} g \|_Q = 0$, that is, $P_V f = P_{V_n} f = \sum_{i=0}^k c_i \eta_{s_i}$, so that the problem is trivial. We consider the case $\mathscr{H}_Q$ is topologically equivalent to the Sobolev space $W_2^{(m)}$ for some positive integer m. This entails that the functions $Q_t^{(\nu)}$, defined by

$$Q_t^{(\nu)}(\cdot) = \frac{\partial^\nu}{\partial s^\nu} Q(s, \cdot) \mid_{s=t},$$

are in $\mathscr{H}_Q$ for $\nu = 0, 1,..., m-1$ and $t \in T$. ($\langle Q_t^{(\nu)}, g \rangle_Q = g^{(\nu)}(t)$). If, for example $g = Q_{t_*}^{(1)} \in \mathscr{H}_Q$ and $T_2 = \{t_*, t_* + \Delta\}$, then it can be shown that $\lim_{\Delta \to 0} \| g - P_{T_2} g \|_Q = 0$, but the limit is not achieved for the points in T_2 distinct .If g has a representation

$$g(t) = \int_0^1 Q(t, s)\, \rho(s)\, ds, \tag{2.2}$$

for some smooth ρ, then the trivial and impossible cases are eliminated. For Q having the continuity properties of a Green's function for a $2m$th order self-adjoint differential operator (typical of spaces topologically equivalent to $W_2^{(m)}$), then (2.2) translates into conditions on $g^{(2m)}$. We assume (2.2).

A sequence $T_n{}^*$, $n = 1, 2,...$ of node sets, $T_{n+1}^* = \{t_{in}^*\}_{i=0}^n$ is said to be asymptotically optimal if

$$\lim_{n \to \infty} \frac{\| g - P_{T_n^*} g \|_Q}{\inf_{T_n} \| g - P_{T_n} g \|_Q} = 1.$$

Let $\mathscr{H}_Q$ be topologically equivalent to $W_2^{(m)}$ and let P_{m,T_n} be the projection $\mathscr{H}_Q$ onto the subspace spanned by

$$\{Q_t^{(\nu)}, t \in T_n, \nu = 0, 1,..., m-1\}.$$

It turns out that it is much easier to find an asymptotically optimal sequence of node sets for $\| g - P_{m,T_n} g \|_Q$ than for $\| g - P_{T_n} g \|_Q$ and these are the results we will present. That is, we find $T_n{}^*$, so that

$$\lim_{n \to \infty} \frac{\| g - P_{m,T_n^*} g \|_Q}{\inf_{T_n} \| g - P_{m,T_n} g \|_Q} = 1. \tag{2.3}$$

The quantity $\| g - P_{m,T_n} g \|_Q$ is the norm of the error committed when using Hermite–Birkhoff data $\{g^{(\nu)}(t), t \in T_n, \nu = 0, 1,..., m-1\}$ in the collocation–projection method to obtain the approximate solution. However, due to the relation

$$\inf_{T_{nm}} \| g - P_{T_{nm}} g \|_Q \leqslant \inf_{T_n} \| g - P_{m,T_n} g \|_Q \leqslant \inf_{T_n} \| g - P_{T_n} g \|_Q, \quad (2.4)$$

asymptotically optimal sequences of node sets with Hermite–Birkhoff data appear to behave like asymptotically optimal sequences of node sets for ordinary data $\{g(t),\ t \in T_n\}$ as well. The right-hand inequality in (2.4) is obvious and furthermore, if $m = 2$ and certain other conditions are fulfilled, it becomes an equality. (See [10, Eq. (13), 14, Theorem 4].) At the infimum, a variational argument gives $\langle g - P_{T_n} g,\ Q_{t_j}^{(1)} \rangle_Q = 0$, $j = 0, 1,..., n$. The left-hand inequality in (2.4) follows since

$$\lim_{\Delta \to 0} \operatorname{span}\{Q_{t+j\Delta},\ j = 0, 1,..., m-1\} = \operatorname{span}\{Q_t^{(\nu)},\ \nu = 0, 1,..., m-1\}.$$

This latter relation is to be interpreted in the sense of strong convergence of the associated projection operators. See also Karlin's improvement theorem [21, 22].

Before stating the most general result on asymptotically optimal sequences of node sets that we know, we briefly describe a special case (see [17] for details) to aid the reader's understanding of the general result.

Let

$$\mathscr{H}_Q = \{ g : g^{(\nu)} \text{ abs cont}, \quad \nu = 0, 1,..., m-1, \quad g^{(\nu)}(0) = 0, \\ \nu = 0, 1,..., m-1, \quad g^{(m)} \in \mathscr{L}_2[0, 1]\},$$

with norm defined by

$$\| g \|_Q^2 = \int_0^1 [a(u)\, g^{(m)}(u)]^2\, du, \quad (2.5)$$

where a is a given strictly positive smooth function. The reproducing kernel Q for $\mathscr{H}_Q$ with norm (2.5) is given by

$$Q(s, t) = \int_0^1 \frac{(s-u)_+^{m-1}}{(m-1)!} \frac{(t-u)_+^{m-1}}{(m-1)!} \frac{du}{a^2(u)}. \quad (2.6)$$

It is shown in [17] that if

$$g(t) = \int_0^1 Q(t, u)\, \rho(u)\, du,$$

then

$$\| g - P_{m,T_n} g \|_Q^2 = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \int_{t_i}^{t_{i+1}} \rho(s)\, B_i(s, t)\, \rho(t)\, ds\, dt, \quad (2.7)$$

where B_i is the Green's function for the operator $(-)^m D^m a^2 D^m$, with boundary conditions $g^{(\nu)}(t_i) = g^{(\nu)}(t_{i+1}) = 0$, $\nu = 0, 1,..., m-1$. Under suitable regularity conditions on a and ρ, for Δ small,

$$\int_{t_i}^{t_{i+1}} \int_{t_i}^{t_{i+1}} \rho(s)\, B_i(s, t)\, \rho(t)\, ds\, dt \approx [\rho(\theta_i)/a(\theta_i)]^2 \int_{t_i}^{t_{i+1}} \int_{t_i}^{t_{i+1}} B_i^0(s, t)\, ds\, dt, \quad (2.8)$$

where $\theta_i \in [t_i, t_{i+1}]$ and B_i^0 is the Green's function B_i for the case $a = 1$ and "$\approx$" means "$=(1 + O(\Delta))$." By [14]

$$\int_{t_i}^{t_{i+1}} \int_{t_i}^{t_{i+1}} B_i^0(s, t)\, ds\, dt = \frac{(m!)^2}{(2m)!\,(2m+1)!} (t_{i+1} - t_i)^{2m+1}.$$

Let h be a strictly positive continuous density on [0, 1], that is, $\int_0^1 h(x)\, dx = 1$ and let the node sets $T_{n+1}(h) = \{t_{in}\}_{i=0}^n$ be defined by

$$\int_0^{t_{in}} h(x)\, dx = i/n, \qquad i = 0, 1,..., n,$$

for $n = 1, 2,...$. Then, using (2.7), (2.8), and $(t_{i+1,n} - t_{in}) \approx 1/nh(\theta_i)$ gives

$$\begin{aligned} \| g - P_{m,T_n(h)} g \|_Q^2 &\approx \frac{(m!)^2}{(2m)!\,(2m+1)!} \frac{1}{n^{2m}} \sum_{i=0}^{n-1} \left[\frac{\rho(\theta_i)}{a(\theta_i)\, h^m(\theta_i)}\right]^2 (t_{i+1,n} - t_{in}), \\ &\approx \frac{(m!)^2}{(2m)!\,(2m+1)!} \frac{1}{n^{2m}} \int_0^1 \left[\frac{\rho(\theta)}{a(\theta)\, h^m(\theta)}\right]^2 d\theta. \end{aligned} \quad (2.9)$$

An asymptotically optimal sequence of node sets according to (2.3) can be found by choosing h to minimize the right-hand side of (2.9). A Holder inequality and the fact that h is a density, gives

$$\int_0^1 \left[\frac{\rho(s)}{a(s)\, h^m(s)}\right]^2 ds \geqslant \left[\int_0^1 \left(\frac{\rho(s)}{a(s)}\right)^{2/(2m+1)} ds\right]^{2m+1}$$

with equality iff $h = h^*$ given by

$$h^*(s) = \frac{(\rho(s)/a(s))^{2/(2m+1)}}{\int_0^1 (\rho(u)/a(u))^{2/(2m+1)}\, du}. \quad (2.10)$$

Then, $T_n(h^*)$, $n = 1, 2,...$ is an asymptotically optimal sequence according to (2.3).

Notice that the parameter function $a(s)$ plays a central role both in the norm on $\mathscr{H}_Q$ and the asymptotically optimal sequence of designs. Thus, the above result should be extendible to range spaces $\mathscr{H}_Q$ topologically equivalent to $W_2^{(m)}$ or a subspace of it, where the "important" part of the norm, that is, the part involving $g^{(m)}$, behaves like (2.5). Modulo the regularity condition (iv) below, this extension is, essentially, the content of the main

THEOREM. [20] *Let Q have a representation*

$$Q(s, t) = \sum_{i=1}^{m} \psi_i(s)\,\psi_i(t) + \int_0^1 \frac{(s-u)_+^{m-1}}{(m-1)!}\,\frac{(t-u)_+^{m-1}}{(m-1)!}\,\frac{du}{a^2(u)}$$

$$+ \int_0^1\int_0^1 \frac{(s-u)_+^{m-1}}{(m-1)!}\,\frac{(t-v)_+^{m-1}}{(m-1)!}\,\frac{A(u,v)}{a(u)\,a(v)}\,du\,dv,$$

where

(i) $\psi_i \in W_2^{(m)}$, $i = 1, 2,\ldots, m$,

(ii) $a > 0$, *a' exists and is bounded,*

(iii) $\int_0^1\int_0^1 A^2(u, v)\,du\,dv < \infty$, *and*

(iv) *the function γ_t given by*

$$\gamma_t(s) = \int_0^s \frac{\partial}{\partial t}\,a(t)\,A(t, \eta)\,\frac{d\eta}{a(\eta)},$$

is well-defined and is in the reproducing kernel Hilbert space with reproducing kernel P given by

$$P(s, t) = \int_0^{\min(s,t)} \frac{du}{a^2(u)} + \int_0^s\int_0^t \frac{A(u, v)}{a(u)\,a(v)}\,du\,dv$$

and

$$\|\gamma_t\|_P \leqslant C < \infty, \qquad t \in [0, 1].$$

Let

$$g(t) = \int_0^1 Q(t, s)\,\rho(s)\,ds,$$

where

(v) $\rho > 0$, *ρ' exists and is bounded.*

Then,

$$\| g - P_{m,T_n(h)}\,g\|_Q^2 = \frac{1}{n^{2m}}\,\frac{(m!)^2}{(2m)!\,(2m+1)!}\int_0^1 \left(\frac{\rho(\theta)}{a(\theta)\,h^m(\theta)}\right)^2 d\theta + o\left(\frac{1}{n^{2m}}\right).$$

Thus, $T_n(h^*)$, $n = 1, 2,\ldots$, with h^* given by (2.10) is an asymptotically optimal sequence of node sets whenever the hypotheses of the main theorem are satisfied.

We remark that if $I + A$ is an invertible operator in $\mathscr{L}_2$, where A is the Hilbert–Schmidt operator with Hilbert–Schmidt kernel $A(s, t)$, then (iv) is equivalent to

$$\int_0^1 \left(\frac{\partial}{\partial t} a(t)\, A(t, \eta)\right)^2 d\eta < M' < \infty.$$

3. Choice of T_n to Minimize $\sup_s |f(s) - f_n(s)|$ when K is a Differential Operator

Let $\eta_t^{(\nu)}$ be the element in $\mathscr{H}_R$ that corresponds to $Q_t^{(\nu)}$ under the isomorphism $\sim$ of (2.1) and let P_{m,V_n} be the projection operator in $\mathscr{H}_R$ onto the subspace spanned by

$$\{\eta_t^{(\nu)}, t \in T_n, \nu = 0, 1,..., m-1\}.$$

We consider only the minimization of $\sup_s |f(s) - P_{m,V_n} f(s)|$ and suppose that $\mathscr{H}_R = V$. Furthermore, most of the discussion of this section is heuristic and we make no attempt to state the weakest regularity conditions under which the results appear to be true.

We have

$$\begin{aligned} f(s) - (P_{m,V_n} f)(s) &= \langle f - P_{m,V_n} f, R_s\rangle_R = \langle f - P_{m,V_n} f, R_s - P_{m,V_n} R_s\rangle_R, \\ &= \langle g - P_{m,T_n} g, \gamma_s - P_{m,T_n}\gamma_s\rangle_Q, \end{aligned}$$

where R_s is the representer of the evaluation functional at s in $\mathscr{H}_R$ and $\gamma_s \in \mathscr{H}_Q$ satisfies $\gamma_s \sim R_s$. Suppose that K is a pth order linear differential operator with a p-dimensional null space. Let $p < r$ and suppose that $\mathscr{H}_R$ is equivalent to the subspace of $W_2^{(r)}$ satisfying the homogenous boundary conditions

$$U_\nu f = 0, \qquad \nu = 1,..., r,$$

$$U_\nu f = \sum_{j=0}^{r-1} \theta_{\nu j} f^{(j)}(0) + \sum_{j=0}^{r-1} \xi_{\nu j} f^{(j)}(1),$$

where the $\{U_\nu\}_{\nu=1}^r$ are linearly independent. (See [19] for an example of the RK.) Here, $m = r - p$. Then, there exists a Green's function $G(s, u)$ with the property

$$f(s) = \int_0^1 G(s, u)\, g(u)\, du,$$

whenever $f \in \mathscr{H}_R \sim g \in \mathscr{H}_Q$, in particular,

$$\eta_t(s) = \int_0^1 G(s, u)\, Q(t, u)\, du.$$

Thus,

$$\gamma_s(t) = \langle \gamma_s , Q_t \rangle_Q = \langle R_s , \eta_t \rangle_R = \eta_t(s) = \int_0^1 Q(t, u)\, \rho_s(u)\, du,$$

where we let

$$\rho_s(u) = G(s, u).$$

If Q is given by (2.6), then it can be shown that

$$\langle g - P_{m,T_n} g, \gamma_s - P_{m,T_n} \gamma_s \rangle_Q = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \int_{t_i}^{t_{i+1}} \rho(u)\, B_i(u, v)\, \rho_s(v)\, du\, dv.$$

If ρ and ρ_s are positive and everything is sufficiently smooth, then by the argument in Section 2,

$$\langle g - P_{m,T_n(h)} g, \gamma_s - P_{m,T_n(h)} \gamma_s \rangle_Q$$

$$\approx \frac{(m!)^2}{(2m)!\,(2m+1)!} \frac{1}{n^{2m}} \sum_{i=0}^{n-1} \frac{\rho(\theta_i)\, \rho_s(\theta_i)}{a^2(\theta_i)\, h^{2m}(\theta_i)} (t_{i+1,n} - t_{in}), \tag{3.1}$$

where $\theta_i \in [t_{in} , t_{n-1,n}]$. Equation (3.1) also holds if the hypotheses of the Theorem hold, but we omit a proof of this assertion.

Now, we discuss the problem of minimizing the supremum over s of the right-hand side of (3.1). The argument below makes no attempt to be rigorous, however, it parallels a rigorous argument in [13], Theorem 4.8], for a special case. Let $\mathscr{M}$ be the class of measures on [0, 1] that assign measure 1 to [0, 1]. Then,

$$\sup_{s\in[0,1]} \langle g - P_{m,T_n(h)} g, \gamma_s - P_{m,T_n(h)} \gamma_s \rangle_Q$$

$$= \sup_{M\in\mathscr{M}} \int_0^1 \langle g - P_{m,T_n(h)} g, \gamma_s - P_{m,T_n(h)} \gamma_s \rangle_Q\, dM(s),$$

$$\approx \frac{(m!)^2}{(2m)!\,(2m+1)!} \frac{1}{n^{2m}} \sup_{M\in\mathscr{M}} \sum_{i=0}^{n-1} \frac{(t_{i+1,n} - t_{in})}{h^{2m}(\theta_i)} \int_0^1 \frac{\rho(\theta_i)\, \rho_s(\theta_i)}{a^2(\theta_i)}\, dM(s),$$

$$\approx \frac{(m!)^2}{(2m)!\,(2m+1)!} \frac{1}{n^{2m}} \sup_{M\in\mathscr{M}} \int_0^1 \frac{d\theta}{h^{2m}(\theta)} \Psi_M{}^2(\theta)\, d\theta, \tag{3.2}$$

where

$$\Psi_M{}^2(\theta) = \int_0^1 \frac{\rho(\theta)\, \rho_s(\theta)}{a^2(\theta)}\, dM(s). \tag{3.3}$$

By a Holder inequality, as before,

$$\int_0^1 \frac{d\theta}{h^{2m}(\theta)} \Psi_M{}^2(\theta)\, d\theta \geqslant \left[\int_0^1 \Psi_M^{2/(2m+1)}(\theta)\, d\theta\right]^{(2m+1)}.$$

with equality iff $h = h_M = \eta_M \Psi_M^{2/(2m+1)}$, where η_M is a constant chosen so that h_M integrates to 1. Thus,

$$\sup_{s\in[0,1]} |f(s) - (P_{m,v_n}f)(s)|$$
$$\geqslant \frac{(m!)}{(2m)!\,(2m+1)!}\frac{1}{n^{2m}} \sup_{M\in\mathcal{M}} \left[\int_0^1 \Psi_M^{2/(2m+1)}(\theta)\,d\theta\right]^{(2m+1)} + o\left(\frac{1}{n^{2m}}\right).$$

The lower bound is then (asymptotically) achieved if $h = h_{M^*} = \eta_{M^*}\Psi_{M^*}^{2/(2m+1)}$, where M^* satisfies

$$\sup_{M\in\mathcal{M}} \int_0^1 \Psi_M^{2/(2m+1)}(\theta)\,d\theta = \int_0^1 \Psi_{M^*}^{2/(2m+1)}(\theta)\,d\theta.$$

Unfortunately, this approach does not appear to be generally attractive computationally, since the $\{\rho_s\}$ are not generally available and even if they were, the problem of finding M^* must be solved. On the other hand, if $g(t)$ is determined *experimentally* for $t \in T_n$ and experimental observations are expensive, then this point of view becomes more attractive. In this case, a preliminary estimate of ρ as given in [1] would be useful.

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